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Ovoids of the Quadric Q(2n, q)

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We consider ovoids of the non-singular quadric $Q(2n, q)$ in $PG(2n, q)$. It is known that $Q(6, q)$ with $q = 2^h$ has no ovoid, while $Q(6, q)$ with $q = 3^h$ admits ovoids. Here we prove that if q is odd, $q \neq 3$, and every ovoid of the non-singular quadric $Q(4, q)$ in $PG(4, q)$ is an elliptic quadric, then $Q(6, q)$, and hence also $Q(2n, q)$ with $n \geqslant 3$, has no ovoid. As a corollary, it follows that $Q(2n, 5)$ and $Q(2n, 7)$, $n \geqslant 3$, have no ovoid.

1. INTRODUCTION

Let $Q(2n, q)$ be a non-singular quadric in the projective space $PG(2n, q)$ for $n \geqslant 2$. (For an introduction to quadrics in projective spaces, see [8, Ch. 22].) An *ovoid* of $Q(2n, q)$ is a set $\mathcal{O}$ of points of $Q(2n, q)$ which has exactly one point in common with each subspace of maximal dimension on $Q(2n, q)$. An ovoid $\mathcal{O}$ of $Q(2n, q)$ satisfies $|\mathcal{O}| = q^n + 1$, see [8, A VI.2.1].

Let $\mathcal{O}$ be an ovoid of $Q(2n, q)$, for $n \geqslant 3$, and let p be a point of $Q(2n, q)$ not on $\mathcal{O}$. Let $p^\perp$ be the tangent hyperplane to $Q(2n, q)$ at p and let $PG(2n-2, q)$ be a hyperplane of $p^\perp$ not containing p. Then the lines of $Q(2n, q)$ through p intersect $PG(2n-2, q)$ in the points of a non-singular quadric $Q(2n-2, q)$ in $PG(2n-2, q)$, see [8, 22.3.8]. The lines of $Q(2n, q)$ through p and a point of $\mathcal{O}$ meet $Q(2n-2, q)$ in the points of an ovoid $\mathcal{O}'$ of $Q(2n-2, q)$ [20]. So we have the following theorem:

THEOREM 1. *If the quadric* $Q(2n, q)$, *for* $n \geqslant 2$, *admits an ovoid, then each quadric* $Q(2m, q)$, *for* $n \geqslant m \geqslant 2$, *admits an ovoid.*

It is well known that an ovoid $\mathcal{O}$ of $Q(4, q)$ defines a translation plane of order q^2 and dimension at most 2 over its kernel [9]. The plane is desarguesian iff $\mathcal{O}$ is contained in a 3-dimensional subspace of $PG(4, q)$; in this case the ovoid $\mathcal{O}$ is an elliptic quadric.

2. A SURVEY OF THE KNOWN EXISTENCE AND NON-EXISTENCE RESULTS

2.1. *Ovoids of* Q(4, q)

Let $Q(4, q)$ be a non-singular quadric in $PG(4, q)$, and let $PG(3, q)$ be a hyperplane of $PG(4, q)$ such that $PG(3, q) \cap Q(4, q)$ is an elliptic quadric $\mathcal{O}$. Then $\mathcal{O}$ is an ovoid of $Q(4, q)$, and an ovoid of $Q(4, q)$ which arises in this way is called *classical*. The connection between ovoids of $Q(4, q)$ and ovoids and spreads of $PG(3, q)$ is discussed in [17, 3.2.1, 3.4.1].

If $q = 2^e$, for e even, then every known ovoid of $Q(4, q)$ is classical. It is straightforward to show that $Q(4, 2)$ admits only classical ovoids. Furthermore, when $q = 4$ or $q = 16$ then every ovoid of $Q(4, q)$ is classical [1, 15] (see [7, 16.1.7]) and [12, 13]. If $q = 2^e$, for e odd and $e \geqslant 3$, then $Q(4, q)$ admits at least one other class of ovoids. The projection of one of these ovoids from the nucleus of $Q(4, q)$ onto a $PG(3, q)$ not containing the nucleus is a Tits ovoid in $PG(3, q)$ [23]. When $q = 8$ or $q = 32$, $Q(4, q)$ admits only these two types of ovoids [6, 14].

Now suppose that q is odd. Then $Q(4,q)$ admits classical ovoids, and further Kantor [9] has constructed two types of non-classical ovoids; type K_1 in the case $q=p^h$, where p is prime and $h>1$ and type K_2 in the case $q=3^{2r+1}$ with $r\geqslant 1$. Recently, Thas and Payne [22] have constructed a third type of non-classical ovoid in $Q(4,q)$ for $q=3^h$ with $h\geqslant 3$. It is known that, for $q=3,5$ or 7, every ovoid of $Q(4,q)$ is classical [2, 11, 5, 3, 4, 10].

2.2. *Ovoids of* $Q(6,q)$

In Thas [20] it is proved that ovoids in $Q(6,q)$, for q even, do not exist.

In the case of q odd, two classes of ovoids of $Q(6,q)$ are known. For $q=3^{2r+1}$ with $r\geqslant 0$ there are the Ree–Tits ovoids [9]; such an ovoid is the set of absolute points of a polarity of the classical generalized hexagon $H(q)$ of order q, $q=3^{2r+1}$, embedded in $Q(6,q)$ [19]. If $r\geqslant 1$, the construction in Section 1 applied to $Q(6,q)$ (for a suitably chosen point p) gives an ovoid of type K_2 of $Q(4,q)$ [9]. For $q=3^h$, where $h\geqslant 1$, there is a second class of ovoids of $Q(6,q)$ [21]. These ovoids can be obtained as follows. Consider the classical generalized hexagon $H(q)$ of order q embedded in $Q(6,q)$, where $q=p^h$. Let $PG(5,q)$ be a hyperplane of $PG(6,q)$ intersecting $Q(6,q)$ in an elliptic quadric $Q^-(5,q)$. Then the lines of $H(q)$ on $Q^-(5,q)$ form a *spread* S of $H(q)$; that is, a set of q^3+1 lines mutually at (maximal) distance 6 in the incidence graph of $H(q)$ [19]. If $q=3^h$ then $H(q)$ admits a duality θ and hence S^θ is an *ovoid* of $H(q)$; that is, a set of q^3+1 points mutually at (maximal) distance 6. By Thas [20], each ovoid of $H(q)$ is an ovoid of $Q(6,q)$ and conversely; so S^θ is an ovoid of $Q(6,q)$.

For each $q=3^{2r+1}$, with $r\geqslant 1$, these two classes of ovoids of $Q(6,q)$ are disjoint [19].

Finally, up to a projectivity, the quadric $Q(6,3)$ has just one ovoid [16, 18].

2.3. *Ovoids of* $Q(2n,q)$ *for* $n>3$

By Section 2.2 and Theorem 1, there is no ovoid in $Q(2n,q)$ for q even and $n>3$.

3. The Main Theorem

In the following, we denote by $\perp$ the polarity of $PG(6,q)$ determined by $Q(6,q)$, for q odd. Thus, for any subspace X of $PG(6,q)$ we will denote by $X^\perp$ the polar space of X with respect to $Q(6,q)$, and we say that each point of X is *orthogonal* to each point of $X^\perp$. For details on this polarity, see [8, Ch. 22]. Furthermore, we will use the letter π to denote a plane of $PG(6,q)$, Σ to denote a 3-dimensional subspace and Π to denote a 4-dimensional subspace of $PG(6,q)$.

Lemma 2. *Let $q\neq 3$ be odd. Suppose that every ovoid of $Q(4,q)$ is an elliptic quadric and that $Q(6,q)$ admits an ovoid $\mathcal{O}$.*
1. *Let L be a line of $Q(6,q)$, external to $\mathcal{O}$. Then the points of $\mathcal{O}$ lying on planes of $Q(6,q)$ about L form an irreducible conic in some plane of $PG(6,q)$.*
2. *If $u_1,u_2,u_3\in\mathcal{O}$ are such that $(u_1u_2u_3)^\perp\cap Q(6,q)$ is a hyperbolic quadric, then $u_1u_2u_3\cap\mathcal{O}$ is an irreducible conic.*

Proof. (1) Let L be a line of $Q(6,q)$, such that $L\cap\mathcal{O}=\phi$. By [8, 22.4.7], there are $q+1$ planes of $Q(6,q)$ through L, all lying in the 4-dimensional space $L^\perp$. Each of these planes contains exactly one point of $\mathcal{O}$; so there arise $q+1$ points $x_1,\ldots,x_{q+1}$ of $\mathcal{O}$.

Let p be a point of Q(6, q) not on $\mathcal{O}$, let $p^\perp$ be the tangent hyperplane of Q(6, q) at p and let PG(4, q) be a hyperplane of $p^\perp$ not containing p. By [20], the lines of Q(6, q) through p and a point of $\mathcal{O}$ meet PG(4, q) in the points of an ovoid $\mathcal{O}'$ of Q(4, q) = Q(6, q) $\cap$ PG(4, q); which, by hypothesis, is an elliptic quadric in a 3-dimensional subspace PG(3, q) of PG(4, q). If we denote the points of $\mathcal{O} \cap p^\perp$ by $y_1, \ldots, y_{q^2+1}$, then the lines $py_1, \ldots, py_{q^2+1}$ are the lines of the quadratic cone $p\mathcal{O}'$ of pPG(3, q) = Π. We will denote this quadratic cone by $\mathscr{C}_p$.

Let $z_i \in L$ and consider the cone $\mathscr{C}_{z_i}$ in the 4-dimensional space Π_i. Now Π_i is distinct from $L^\perp$ (as Π_i contains points of $\mathcal{O}$ not in $L^\perp$), so $\Pi_i \cap L^\perp$ is a 3-dimensional space Σ_i on z_i and hence $C_{z_i} \cap L^\perp$ is a quadratic cone K_i in Σ_i . Since $x_1, \ldots, x_{q+1}$ are the only points of $\mathcal{O}$ in $L^\perp$, so $K_i = \{z_ix_1, \ldots, z_ix_{q+1}\}$.

Now, $\Sigma_i \neq \Sigma_j$ for $i \neq j$, for otherwise $z_iz_j = L \subseteq \Sigma_i \subseteq \Pi_i$, implying that $L \cap \mathcal{O} \neq \phi$ (as Π_i meets Q(6, q) in a set of lines on z_i each containing a point of $\mathcal{O}$), a contradiction. Now Σ_i and Σ_j both lie in $L^\perp$; so for $i \neq j$, $\Sigma_i \cap \Sigma_j$ is a plane PG(2, q). So PG(2, q) contains the points $x_1, \ldots, x_{q+1}$, which lie on both the quadratic cones K_i and K_j; hence $\{x_1, \ldots, x_{q+1}\}$ must be the points of an irreducible conic in PG(2, q).

(2) Let $u_1, u_2, u_3 \in \mathcal{O}$ be such that $(u_1u_2u_3)^\perp \cap$ Q(6, q) is a hyperbolic quadric. A line M of this hyperbolic quadric is external to $\mathcal{O}$, for otherwise the plane u_1M of Q(6, q) contains more than one point of $\mathcal{O}$. By part (1) of this lemma, the $q+1$ points of $\mathcal{O}$ on planes of Q(6, q) about M (including u_1, u_2, u_3) form an irreducible conic in some plane of PG(6, q); hence $u_1u_2u_3 \cap \mathcal{O}$ is an irreducible conic. □

THEOREM 3. *If every ovoid of* Q(4, q), *where q is odd and $q \neq 3$, is an elliptic quadric then* Q(6, q) *has no ovoid.*

PROOF. Suppose that the only ovoids of Q(4, q), for q odd and $q \neq 3$, are the elliptic quadrics. Assume, by way of contradiction, that Q(6, q) admits an ovoid $\mathcal{O}$. The main effort in the proof is to show that a 3-dimensional space meeting $\mathcal{O}$ in a conic, arising as in Lemma 2(1), and a further point must contain at least $(q^2 - q + 6)/2$ points of $\mathcal{O}$.

We first consider some particular subspaces of PG(6, q) and determine their intersections with Q(6, q). Let $\mathscr{C}$ be a conic contained in $\mathcal{O}$, arising as in Lemma 2(1) from a line L of Q(6, q) with $L \cap \mathcal{O} = \phi$. Let π be the plane containing $\mathscr{C}$. Since $\pi \cap \mathcal{O} = \mathscr{C} \subseteq$ Q(6, q), it follows that $\pi \cap$ Q(6, q) = $\mathscr{C}$. Hence $\pi^\perp$ is a 3-dimensional space meeting Q(6, q) in either an elliptic or a hyperbolic quadric [8, 22.7.2]. But $\pi \subseteq L^\perp$; hence $L \subseteq \pi^\perp$, so that $\pi^\perp \cap$ Q(6, q) = H is a hyperbolic quadric. Next, let $x \in \mathcal{O} \backslash \mathscr{C}$, and let $\Sigma = x\pi$. Let $x' \in \mathscr{C}$. Since xx' is a line meeting Q(6, q) in $\{x, x'\}$, the space $(xx')^\perp$ is a 4-dimensional space PG(4, q) such that PG(4, q) $\cap$ Q(6, q) = Q(4, q) is a non-singular quadric [8, 22.7.2]. Finally, we note that PG(4, q) $\cap\, \pi^\perp = (xx')^\perp \cap \pi^\perp = (x\pi)^\perp = \Sigma^\perp = \pi'$ is a plane. Since $\Sigma \cap$ Q(6, q) contains an irreducible conic and a further point, it is either a hyperbolic quadric, an elliptic quadric or a quadratic cone. But $\Sigma \cap$ Q(6, q) cannot be a quadratic cone; for otherwise x would lie on a line of Q(6, q) with at least one point on $\mathscr{C}$, which is impossible since $\{x\} \cup \mathscr{C} \subseteq \mathcal{O}$. Thus $\Sigma \cap$ Q(6, q) is either an elliptic or a hyperbolic quadric; so by [8, 22.7.2] it follows that $\pi' \cap$ Q(6, q) = $\mathscr{C}'$ is an irreducible conic.

We are interested in counting the points $x'' \in \mathscr{C} \backslash \{x'\}$ such that $(xx'x'')^\perp \cap$ Q(6, q) is a hyperbolic quadric (for then we can apply Lemma 2(2) to show that the plane $xx'x''$ contains $q+1$ points of $\mathcal{O}$). Let $x'' \in \mathscr{C} \backslash \{x'\}$ be such that $(xx'x'')^\perp \cap$ Q(6, q) is a hyperbolic quadric. Since $xx'x'' \subseteq \Sigma$, it follows that $\mathscr{C}' = \Sigma^\perp \cap$ Q(6, q) $\subseteq (xx'x'')^\perp \cap$ Q(6, q). Similarly, $xx' \subseteq xx'x''$ so $(xx'x'')^\perp \cap Q(6, q) \subseteq (xx')^\perp \cap$ Q(6, q) = Q(4, q). Hence such a hyperbolic quadric contains $\mathscr{C}'$ and is contained in Q(4, q).

We count the number of 3-dimensional sections of Q(4, q) which contain $\mathscr{C}'$ and which are hyperbolic quadrics. First, we show that no 3-dimensional section of $Q(4,q)$ containing $\mathscr{C}'$ can be a quadratic cone, by showing that $(\pi')^\perp$ contains no point of $Q(4,q)$. Suppose, by way of contradiction, that $(\pi')^\perp = \Sigma$ contains a point u of $Q(4,q) = (xx')^\perp \cap Q(6,q)$. It follows that u is orthogonal to each point of the line xx' and, in particular, $u \notin \mathscr{C}$ since $u, x \in \mathcal{O}$. Now $x'u$ is a line of Σ and of $Q(6,q)$; so $\Sigma \cap Q(6,q)$ is a quadratic cone or a hyperbolic quadric. In either case, since $\mathscr{C} \subseteq \Sigma \cap Q(6,q)$, x is orthogonal to at least one point of $\mathscr{C}$, a contradiction as $\{x\} \cup \mathscr{C} \subseteq \mathcal{O}$. We have shown that every 3-dimensional section of $Q(4,q)$ containing $\mathscr{C}'$ and a line of $Q(4,q)$ is a hyperbolic quadric. Let p be a point of $\mathscr{C}'$. Such a 3-dimensional section must contain a line of $Q(4,q)$ on p, so the number of 3-dimensional sections of $Q(4,q)$ containing $\mathscr{C}'$ and which are hyperbolic quadrics is just the number of generators of $Q(4,q)$ through p divided by the number of generators of such a hyperbolic quadric through p. This number is therefore $(q+1)/2$.

We will show that all but exactly one of these $(q+1)/2$ sections arise as in Lemma 2(2) from a point $x'' \in \mathscr{C}$. Let Σ' be a 3-dimensional space, meeting $Q(4,q)$ in a hyperbolic quadric on $\mathscr{C}'$. Since $\pi' \subseteq \Sigma' \subseteq PG(4,q)$, the space $(\Sigma')^\perp$ is a plane in Σ through xx'. Now $(\Sigma')^\perp$ contains a point $x'' \in \mathscr{C}\backslash\{x'\}$ provided that it does not meet π in the tangent N to $\mathscr{C}$ at x'. Next, we show that the 3-dimensional space $(xN)^\perp$ does occur in the $(q+1)/2$ 3-dimensional spaces counted above. First, $N^\perp$ is a 4-dimensional space containing x' and $\pi^\perp$ (as $\{x'\} = N \cap \mathscr{C}$ and $N \subseteq \pi$); so $N^\perp = x'\pi^\perp$ (note that $x' \notin \pi^\perp$). Hence, $(xN)^\perp = (xx'N)^\perp = N^\perp \cap (xx')^\perp = x'\pi^\perp \cap PG(4,q)$. Next we consider $(xN)^\perp \cap Q(4,q) = x'\pi^\perp \cap PG(4,q) \cap Q(4,q) = (x'\pi^\perp \cap Q(6,q)) \cap PG(4,q)$. Now $x'\pi^\perp \cap Q(6,q) = N^\perp \cap Q(6,q)$ is either an elliptic quadric cone or a hyperbolic quadric cone [8, 22.7.2]. Since $\pi^\perp \cap Q(6,q) = H$ is a hyperbolic quadric, it follows that $x'\pi^\perp \cap Q(6,q) = x'H$ is a hyperbolic quadric cone. Now both $x'\pi^\perp$ and $PG(4,q)$ are contained in $(x')^\perp$ (as x' is contained in each of π and $PG(4,q)^\perp = xx'$) and $x' \notin PG(4,q)$ [8, 22.3.6 Cor]. Hence $x'\pi^\perp \cap PG(4,q)$ is a 3-dimensional space not containing x' and therefore is such that $[x'\pi^\perp \cap PG(4,q)] \cap Q(4,q)$ is a hyperbolic quadric, as required. Thus $(xN)^\perp$ is a 3-dimensional subspace of $PG(4,q)$ containing π' and meeting $Q(4,q)$ in a hyperbolic quadric, but which does not arise as in Lemma 2(2) from a point x'' of $\mathscr{C}\backslash\{x'\}$.

Consequently, there are exactly $(q-1)/2$ points $x'' \in \mathscr{C}\backslash\{x'\}$ such that $(xx'x'')^\perp \cap Q(6,q)$ is a hyperbolic quadric. We denote these points by x_i'' for $i = 1, \ldots, (q-1)/2$. By Lemma 2(2), $xx'x_i'' \cap \mathcal{O} = \mathscr{C}_i$ is an irreducible conic for $i = 1, \ldots, (q-1)/2$.

Now $\Sigma \cap Q(6,q)$ is a hyperbolic quadric, an elliptic quadric or a quadratic cone. Suppose that it is a hyperbolic quadric or a quadratic cone. Then x is orthogonal to at least one point of $\mathscr{C}$, a contradiction since $x \in \mathcal{O}$ and $\mathscr{C} \subseteq \mathcal{O}$. Thus $\Sigma \cap Q(6,q) = Q^-(3,q)$ is an elliptic quadric containing $xx'x_i'' \cap \mathcal{O}$ for $i = 1, \ldots, (q-1)/2$; so $Q^-(3,q)$ contains at least $q + 1 + (q-1)(q-2)/2 + 1 = (q^2 - q + 6)/2$ points of $\mathcal{O}$.

Let $PG(5,q)$ be a hyperplane of $PG(6,q)$ containing Σ and such that $PG(5,q) \cap Q(6,q) = Q^+(5,q)$ is a hyperbolic quadric. Since each plane of $Q^+(5,q)$ contains a unique point of $\mathcal{O}$, and using [8, 22.4.7], we see that $Q^+(5,q) \cap \mathcal{O} = \mathcal{O}'$ satisfies $|\mathcal{O}'| = q^2 + 1$.

We now show that $\mathcal{O}' = Q^-(3,q)$. Suppose, to the contrary, that there exists a point $z \in \mathcal{O}'\backslash Q^-(3,q)$ and let $\Sigma'' = z\pi$. Assume further that there exists a point $z' \in \mathcal{O}'\backslash(\Sigma \cup \Sigma'')$ and let $\Sigma''' = z'\pi$. Note that Σ, Σ'' and Σ''' are distinct 3-dimensional subspaces of $PG(5,q)$ each containing $\mathscr{C}$ and each containing a further point of $\mathcal{O}$, so the above arguments show that each contains at least $(q^2 - q + 6)/2$ points of $\mathcal{O}$, which are also points of $\mathcal{O}'$. Hence the union $\Sigma \cup \Sigma'' \cup \Sigma'''$ contains at least $3((q^2 - q + 6)/2 - (q + 1)) + (q+1) = (3q^2 - 7q + 14)/2$ points of $\mathcal{O}'$. Hence $(3q^2 - 7q + 14)/2 \leq q^2 + 1$, implying that $q \in \{3, 4\}$, contrary to the hypothesis that $q \neq 3$ is odd. Still assuming that there exists a point $z \in \mathcal{O}' \backslash Q^-(3,q)$, we have shown that $\mathcal{O}'$ is contained in $\Sigma \cup \Sigma''$. Since $\Sigma \cap Q^+(5,q) = \Sigma \cap Q(6,q) = Q^-(3,q)$, it follows that the 4-dimensional space $\Sigma\Sigma''$ meets $Q^+(5,q)$ in a non-singular quadric $Q(4,q)'$. Thus $\mathcal{O}'$ is a set of q^2+1 points of $Q(4,q)'$ such that no line of $Q(4,q)'$ contains more than one point of $\mathcal{O}'$ (else there is a plane of $Q^+(5,q)$ containing more than one point of $\mathcal{O}'$). By the initial assumption, every ovoid of $Q(4,q)'$ is an elliptic quadric; so $\mathcal{O}'$ is an elliptic quadric sharing x and $\mathscr{C}$ with $Q^-(3,q)$. Hence $\mathcal{O}'$ and $Q^-(3,q)$ are contained in the same 3-dimensional space, implying that $\mathcal{O}' = Q^-(3,q)$, and so contradicting the existence of $z \in \mathcal{O}' \backslash Q^-(3,q)$. It follows that there does not exist a point $z \in \mathcal{O}' \backslash Q^-(3,q)$; so $\mathcal{O}' = Q^-(3,q)$.

Let $w \in \mathcal{O} \backslash Q^-(3,q)$. Since $\Sigma \cap Q(6,q) = Q^-(3,q)$ and w is not orthogonal to any point of $Q^-(3,q)$, the 4-dimensional space $w\Sigma$ meets $Q(6,q)$ in a non-singular quadric $Q(4,q)''$ containing $Q^-(3,q)$. Thus $Q(4,q)''$ contains at least q^2+2 points of $\mathcal{O}$; a contradiction since $|Q(4,q)'' \cap \mathcal{O}| \leqslant q^2+1$ (as no line of $Q(4,q)''$ contains more than 1 point of $\mathcal{O}$; so $|Q(4,q)'' \cap \mathcal{O}|$ cannot have more points than an ovoid of $Q(4,q)''$). Hence $\mathcal{O} = Q^-(3,q)$, which is impossible as $|\mathcal{O}| = q^3+1$, while $|Q^-(3,q)| = q^2+1$.

We conclude, therefore, that $Q(6,q)$ has no ovoid. □

COROLLARY 4. *The non-singular quadrics* $Q(6,5)$ *and* $Q(6,7)$ *have no ovoid.*

PROOF. In Section 2.1 we mentioned that $Q(4,q)$, for $q \in \{5,7\}$, admits only classical ovoids. Hence $Q(6,q)$ for $q \in \{5,7\}$ has no ovoid. □

CONJECTURE 5. The non-singular quadric $Q(6,q)$, for $q \neq 3^h$, has no ovoid.

COROLLARY 6. *The non-singular quadrics* $Q(2n,5)$ *and* $Q(2n,7)$, *for* $n \geqslant 3$, *have no ovoid.*

PROOF. The proof follows from Theorem 1 and Corollary 4. □

COROLLARY 7. *The generalized hexagons* H(5) *and* H(7) *have no ovoid.*

PROOF. The proof follows from Corollary 4 and Section 2.2. □

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